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LETTER TO THE EDITOR

Periodic multiwave solutions to the Toda-type cellular automaton

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Abstract
An ultradiscretization of the Riemann theta function is proposed. The ultradiscretization satisfies an addition formula, which is an ultradiscrete analogue of an addition formula for the Riemann theta function. Using the addition formula, periodic multiwave solutions to the Toda-type cellular automaton are obtained.

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1. Introduction

The discrete two-dimensional Toda (2D d-Toda) equation proposed by Hirota [1] is a full discrete version of the 2D Toda lattice equation. The bilinear form, which is equivalent to the discrete KP equation, is given as follows:

$$\frac{f_{n,m}^{t+1}f_{n,m}^{t-1} - f_{n,m+1}^{t}f_{n,m-1}^{t}}{\delta^2} = f_{n+1,m}^{t}f_{n-1,m}^{t} - f_{n,m+1}^{t}f_{n,m-1}^{t} \tag{1}$$

where δ is a positive number. The bilinear form (1) is reduced to the one of the 2D Toda lattice equation in the continuous limit $\delta \to +0$. In 1990, Takahashi and Satsuma proposed an important ultradiscrete integrable system called the soliton cellular automaton (SCA) [2]. In 1996, a general method to obtain the SCA from discrete soliton equations was proposed [3]. This procedure, which is called the ultradiscretization, has been applied to various kinds of soliton equations [4, 5]. In this context, the 2D d-Toda equation was ultradiscretized and the 2D Toda-type CA (TTCA) was obtained [6]. Introduce a new variable $\rho_{n,m}^t$ and a parameter L through

$$f_{n,m}^t = \mathrm{e}^{\frac{\rho_{n,m}^t}{\varepsilon}} \qquad \delta^2 = \mathrm{e}^{-\frac{L}{\varepsilon}}$$

where ε is a positive number. Substitute these into (1) and take the limit $\varepsilon \to +0$, then the bilinear form of the 2D TTCA is obtained:

$$\rho_{n,m}^{t+1} + \rho_{n,m}^{t-1} = \max\left[\rho_{n,m+1}^t + \rho_{n,m-1}^t, \rho_{n+1,m}^t + \rho_{n-1,m}^t - L\right] \tag{2}$$

where we use an identity for $A, B \in \mathbb{R}$

$$\lim_{\varepsilon\to+0} \varepsilon \log\left(e^{\frac{A}{\varepsilon}} + e^{\frac{B}{\varepsilon}}\right) = \max[A, B].$$

By introducing a variable $U^t_{n,m}$ through

$$U^t_{n,m} = \rho^t_{n+1,m} + \rho^t_{n-1,m} - \rho^t_{n,m+1} - \rho^t_{n,m-1} \tag{3}$$

the evolution rule of the 2D TTCA is reduced

$$\begin{aligned} U^{t+1}_{n,m} + U^{t-1}_{n,m} - U^t_{n,m+1} - U^t_{n,m-1} &= \max\left[0, U^t_{n+1,m} - L\right] + \max\left[0, U^t_{n-1,m} - L\right] \\ &\quad - \max\left[0, U^t_{n,m+1} - L\right] - \max\left[0, U^t_{n,m-1} - L\right]. \end{aligned}$$

The variable $U^t_{n,m}$ stands for the value of the cell at position (n, m) and time t.

Remark 1. The 1D TTCA [4] is reduced from the 2D TTCA by eliminating the independent variable m.

It is known that the 2D TTCA has the N-soliton solution, which is an ultradiscretization of the one to the 2D d-Toda equation [6]. On the other hand, it is also known that the 2D d-Toda equation has algebro-geometric solutions expressed by the Riemann theta function $\vartheta(z, \Omega)$, whose degeneration is the N-soliton solution [7]. However, as far as the author knows, such (quasi-) periodic multiwave solutions to the 2D TTCA have not been obtained yet.

In [8], we propose an ultradiscretization of elliptic theta function $\vartheta_{00}(z, \tau)$, which is denoted by $\Theta_0(x)$, and obtain an addition formula. The addition formula is reduced from an ultradiscrete analogue of Riemann's theta formula. Using the addition formula, periodic single wave solutions to the 1D TTCA and the box-ball system [5, 9, 10] are obtained. In this letter, we propose an ultradiscretization of $\vartheta(z, \Omega)$ and derive an addition formula from an ultradiscrete analogue of the generalized Riemann theta formula. We show that we can obtain periodic multiwave solutions to the 2D TTCA by using the addition formula. Note that addition properties for $\vartheta(z, \Omega)$ play an important role in the study of discrete integrable systems [11, 12]. The solutions thus obtained can be directly connected to the one to the 2D d-Toda equation through the ultradiscretization.

2. Ultradiscrete Riemann theta function

The Riemann theta function $\vartheta(z, \Omega)$ in g variables is defined as follows [13]:

$$\vartheta(z, \Omega) := \sum_{n\in\mathbb{Z}^g} e^{\pi i\langle n, \Omega n\rangle}\, e^{2\pi i\langle n, z\rangle}$$

where $z \in \mathbb{C}^g$, Ω is a symmetric $g \times g$ complex matrix whose imaginary part is positive definite and $\langle , \rangle : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ is the standard inner product. In this letter, we propose an ultradiscretization of $\vartheta(z, \Omega)$ for general $g \in \mathbb{N}$.

Note a functional equation

$$\vartheta(z, \Omega) = \det\left(\frac{\Omega}{i}\right)^{-1/2} e^{-\pi i\langle z, \Omega^{-1} z\rangle}\, \vartheta(\Omega^{-1} z, -\Omega^{-1}).$$

Let $\Omega = i\pi\varepsilon B$, where B is a positive definite and symmetric $g \times g$ real matrix and ε is a positive number. Then we obtain

$$\vartheta(z, \Omega) = \det(\pi\varepsilon B)^{-\frac{1}{2}}\, e^{-\frac{1}{\varepsilon}\langle z, B^{-1} z\rangle} \sum_{n\in\mathbb{Z}^g} e^{-\frac{1}{\varepsilon}\langle n, B^{-1} n\rangle}\, e^{\frac{2}{\varepsilon}\langle n, B^{-1} z\rangle}.$$

Assume $\mathrm{Im}\, \boldsymbol{z} = 0$ then the function always takes positive values. Hence, we can ultradiscretize $\vartheta(\boldsymbol{z}, \Omega)$ on $\mathbb{R}^g$. Actually, we have

$$\lim_{\varepsilon\to+0} \varepsilon \log^\dagger \vartheta(\boldsymbol{z}, \Omega)|_{\mathrm{Im}\, \boldsymbol{z}=0} = -\langle \mathrm{Re}\, \boldsymbol{z}, B^{-1}\, \mathrm{Re}\, \boldsymbol{z}\rangle + \max_{\boldsymbol{n}\in\mathbb{Z}^g}[\langle \boldsymbol{n}, B^{-1}(2\, \mathrm{Re}\, \boldsymbol{z} - \boldsymbol{n})\rangle]$$

where we assume that $\log^\dagger f_1 f_2 \cdots f_n := \mathrm{Log} f_1 + \mathrm{Log} f_2 + \cdots + \mathrm{Log} f_n$ and $\mathrm{Log} f$ stands for the principal value of $\log f$. We denote this by $\Theta(\boldsymbol{x}; B)$ and call the ultradiscrete Riemann theta function. The periodicity of the original function survives trough the ultradiscretization; $\Theta(\boldsymbol{x}; B)$ has the following periodicity:

$$\Theta(\boldsymbol{x} + \boldsymbol{m}; B) = \Theta(\boldsymbol{x}; B) \qquad \text{for} \quad {}^\forall \boldsymbol{m} \in \mathbb{Z}^g.$$

Remark 2. Let B be a diagonal matrix then $\Theta(\boldsymbol{x}; B)$ is expressed by the sum of the ultradiscrete elliptic theta function $\Theta_0(x)$:

$$\Theta(\boldsymbol{x}; B) = \sum_{j=1}^{g} \frac{1}{B_{jj}} \Theta_0(x_j)$$

where we assume B_{jj} $(j = 1, 2, \ldots, g)$ to be the jth diagonal element of B and x_j $(j = 1, 2, \ldots, g)$ the jth element of $\boldsymbol{x}$.

The ultradiscrete Riemann theta function $\Theta(\boldsymbol{x}; B)$ satisfies an addition formula, which is an ultradiscrete analogue of the following addition formula for $\vartheta(\boldsymbol{z}, \Omega)$:

$$\begin{aligned}\vartheta(\boldsymbol{x} + \boldsymbol{u}, \Omega)\vartheta(\boldsymbol{x} - \boldsymbol{u}, \Omega)\vartheta(0, \Omega)^2 \\ = \frac{1}{2^g} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \mathrm{e}^{4\pi\mathrm{i}\langle\boldsymbol{\alpha},\Omega\boldsymbol{\alpha}+\boldsymbol{x}+\boldsymbol{u}\rangle} \vartheta(\boldsymbol{x} + \Omega\boldsymbol{\alpha} + \boldsymbol{\beta}, \Omega)^2 \vartheta(\boldsymbol{u} + \Omega\boldsymbol{\alpha} + \boldsymbol{\beta}, \Omega)^2. \end{aligned} \qquad (4)$$

In order to derive the addition formula for $\Theta(\boldsymbol{x}; B)$, we first show an ultradiscrete analogue of the generalized Riemann theta formula.

Theorem 1. *Let*

$$(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4) := \frac{1}{2} (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Then, for $\boldsymbol{x}_i \in \mathbb{R}^g$ $(i = 1, 2, 3, 4)$*, the following identity holds:*

$$\max_{\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \left[\sum_{i=1}^{4} \Theta(\boldsymbol{x}_i + \boldsymbol{\beta}; B) \right] = \max_{\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \left[\sum_{i=1}^{4} \Theta(\boldsymbol{y}_i + \boldsymbol{\beta}; B) \right]. \qquad (5)$$

Proof. By the definition of $\Theta(\boldsymbol{x}; B)$, equation (5) is reduce to the following:

$$\begin{aligned}\max_{\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \left[\sum_{i=1}^{4} \max_{\boldsymbol{n}_i\in\mathbb{Z}^g} [\langle \boldsymbol{n}_i - \boldsymbol{\beta}, B^{-1}(2\boldsymbol{x}_i - (\boldsymbol{n}_i - \boldsymbol{\beta}))\rangle] \right] \\ = \max_{\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \left[\sum_{i=1}^{4} \max_{\boldsymbol{n}_i\in\mathbb{Z}^g} [\langle \boldsymbol{n}_i - \boldsymbol{\beta}, B^{-1}(2\boldsymbol{y}_i - (\boldsymbol{n}_i - \boldsymbol{\beta}))\rangle] \right]. \end{aligned} \qquad (6)$$

Noting an identity

$$\sum_{i=1}^{4} \max_{\boldsymbol{n}_i \in \mathbb{Z}^g} [\langle \boldsymbol{n}_i - \boldsymbol{\beta}, B^{-1}(2\boldsymbol{x}_i - (\boldsymbol{n}_i - \boldsymbol{\beta}))\rangle] = \max_{\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3,\boldsymbol{n}_4 \in \mathbb{Z}^g} \left[\sum_{i=1}^{4} \langle \boldsymbol{n}_i - \boldsymbol{\beta}, B^{-1}\,(2\boldsymbol{x}_i - (\boldsymbol{n}_i - \boldsymbol{\beta}))\rangle \right]$$

we have

$$\text{lhs of (6)} = \max_{\beta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \left[\max_{\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3,\boldsymbol{n}_4 \in \mathbb{Z}^g} \left[\sum_{i=1}^{4} \langle \boldsymbol{n}_i - \boldsymbol{\beta}, B^{-1}\,(2\boldsymbol{x}_i - (\boldsymbol{n}_i - \boldsymbol{\beta}))\rangle \right]\right].$$

Let

$$(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3, \boldsymbol{m}_4) := (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3, \boldsymbol{n}_4)\, T$$

where we put

$$T := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that

$$T\mathbb{Z}^4 = \left\{ (a_1, a_2, a_3, a_4) \,\middle|\, a_j \in \frac{1}{2}\mathbb{Z}, a_j + a_k \in \mathbb{Z}, \sum_{j=1}^{4} a_j \in 2\mathbb{Z} \right\}$$

so that coset representatives for $T\mathbb{Z}^4 \cap \mathbb{Z}^4$ in $T\mathbb{Z}^4$ are $(0, 0, 0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Also note the following identities:

$$\sum_{i=1}^{4} \langle \boldsymbol{n}_i, B^{-1}\boldsymbol{n}_i\rangle = \sum_{i=1}^{4} \langle \boldsymbol{m}_i, B^{-1}\boldsymbol{m}_i\rangle \qquad \sum_{i=1}^{4} \langle \boldsymbol{n}_i, B^{-1}\boldsymbol{x}_i\rangle = \sum_{i=1}^{4} \langle \boldsymbol{m}_i, B^{-1}\boldsymbol{y}_i\rangle$$

we have

$$\begin{aligned} \text{lhs of (6)} &= \max_{\beta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \left[\max_{\boldsymbol{m}_1,\boldsymbol{m}_2,\boldsymbol{m}_3,\boldsymbol{m}_4 \in \mathbb{Z}^g} \left[\sum_{i=1}^{4} \langle \boldsymbol{m}_i - \boldsymbol{\beta}, B^{-1}(2\boldsymbol{y}_i - (\boldsymbol{m}_i - \boldsymbol{\beta}))\rangle \right]\right] \\ &= \text{rhs of (6)}. \end{aligned}$$

This completes the proof. □

By setting $\boldsymbol{x} := \boldsymbol{x}_1 = \boldsymbol{x}_2$ and $\boldsymbol{u} := \boldsymbol{x}_3 = \boldsymbol{x}_4$ in (5), we get

$$2 \max_{\beta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} [\Theta(\boldsymbol{x} + \boldsymbol{\beta}; B) + \Theta(\boldsymbol{u} + \boldsymbol{\beta}; B)] = \max_{\beta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} [\Theta(\boldsymbol{x} + \boldsymbol{u} + \boldsymbol{\beta}; B) + \Theta(\boldsymbol{x} - \boldsymbol{u} + \boldsymbol{\beta}; B) + 2\Theta(\boldsymbol{\beta}; B)].$$

Noting an inequality for $\boldsymbol{x}$

$$\Theta(\boldsymbol{x} + \boldsymbol{\beta}; B) + \Theta(\boldsymbol{\beta}; B) \leqslant \Theta(\boldsymbol{x}; B) \qquad \text{for} \quad {}^\forall\boldsymbol{\beta} \in \tfrac{1}{2}\mathbb{Z}^g\big/\mathbb{Z}^g$$

we obtain the following addition formula.

Corollary 1 (Addition formula). *For $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^g$, the following identity holds:*

$$\Theta(\boldsymbol{x}+\boldsymbol{u}; B) + \Theta(\boldsymbol{x}-\boldsymbol{u}; B) = 2 \max_{\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} [\Theta(\boldsymbol{x}+\boldsymbol{\beta}; B) + \Theta(\boldsymbol{u}+\boldsymbol{\beta}; B)]\,. \tag{7}$$

3. Solutions to the 2D TTCA

Now we construct periodic multiwave solutions to the 2D TTCA. Let

$$\rho^t_{n,m} = \Theta(\boldsymbol{x}; B) + \langle \boldsymbol{x}, B^{-1}\boldsymbol{x}\rangle \qquad \boldsymbol{x} = \boldsymbol{u}n + \boldsymbol{v}t + \boldsymbol{w}m. \tag{8}$$

We consider two kinds of dispersion relations, one for single wave and the other for multiwave. For single wave, we assume that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathbb{R}^g$ satisfy the following dispersion relations:

$$\Theta(\boldsymbol{v}; B) + \langle \boldsymbol{v}, B^{-1}\boldsymbol{v}\rangle = 0 \tag{9}$$

$$\Theta(\boldsymbol{w}; B) + \langle \boldsymbol{w}, B^{-1}\boldsymbol{w}\rangle = 0 \tag{10}$$

$$\Theta(\boldsymbol{w}; B) - \Theta\left(\boldsymbol{w}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) > \Theta(\boldsymbol{v}; B) - \Theta\left(\boldsymbol{v}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) > \Theta(\boldsymbol{u}; B) - \Theta\left(\boldsymbol{u}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) \tag{11}$$

$$\max\left[\Theta(\boldsymbol{\alpha}; B) + \Theta(\boldsymbol{u}; B), \Theta\left(\tfrac{\boldsymbol{1}}{\boldsymbol{2}}+\boldsymbol{\alpha}; B\right) + \Theta\left(\tfrac{\boldsymbol{1}}{\boldsymbol{2}}+\boldsymbol{u}; B\right)\right] \geqslant \Theta(\boldsymbol{u}+\boldsymbol{\alpha}; B)$$
$$\text{for } 0, \tfrac{\boldsymbol{1}}{\boldsymbol{2}} \neq \boldsymbol{\alpha} \in \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \tag{12}$$

$$\max\left[\Theta(\boldsymbol{\alpha}; B) + \Theta(\boldsymbol{v}; B), \Theta\left(\tfrac{\boldsymbol{1}}{\boldsymbol{2}}+\boldsymbol{\alpha}; B\right) + \Theta\left(\tfrac{\boldsymbol{1}}{\boldsymbol{2}}+\boldsymbol{v}; B\right)\right] \geqslant \Theta(\boldsymbol{v}+\boldsymbol{\alpha}; B)$$
$$\text{for } 0, \tfrac{\boldsymbol{1}}{\boldsymbol{2}} \neq \boldsymbol{\alpha} \in \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \tag{13}$$

$$\max\left[\Theta(\boldsymbol{\alpha}; B) + \Theta(\boldsymbol{w}; B), \Theta\left(\tfrac{\boldsymbol{1}}{\boldsymbol{2}}+\boldsymbol{\alpha}; B\right) + \Theta\left(\tfrac{\boldsymbol{1}}{\boldsymbol{2}}+\boldsymbol{w}; B\right)\right] \geqslant \Theta(\boldsymbol{w}+\boldsymbol{\alpha}; B)$$
$$\text{for } 0, \tfrac{\boldsymbol{1}}{\boldsymbol{2}} \neq \boldsymbol{\alpha} \in \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \tag{14}$$

where $\frac{\boldsymbol{1}}{\boldsymbol{2}}$ stands for a g-vector ${}^t\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \in \mathbb{R}^g$. For multiwave, we assume the following:

$$\langle \boldsymbol{w}, B^{-1}\boldsymbol{w}\rangle \leqslant \langle \boldsymbol{u}, B^{-1}\boldsymbol{v}\rangle \tag{15}$$

$$\langle \boldsymbol{w}, B^{-1}\boldsymbol{w}\rangle \leqslant \langle \boldsymbol{v}, B^{-1}\boldsymbol{v}\rangle \tag{16}$$

$$\boldsymbol{u} \equiv \pm\boldsymbol{v} \equiv \pm\boldsymbol{w} \quad (\text{mod } \mathbb{Z}^g). \tag{17}$$

We assume that the parameter L in (2) is given as follows:

$$L = 2\left(\langle \boldsymbol{u}, B^{-1}\boldsymbol{u}\rangle - \langle \boldsymbol{v}, B^{-1}\boldsymbol{v}\rangle + \Theta\left(\boldsymbol{u}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) - \Theta\left(\boldsymbol{v}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right)\right). \tag{18}$$

Substitute (8) into the bilinear form of the 2D TTCA (2). Then, by using the addition formula (7), the left-hand side of (2) is reduced to

$$\begin{aligned}
\rho^{t+1}_{n,m} + \rho^{t-1}_{n,m} &= 2 \max_{\boldsymbol{\beta}\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} [\Theta(\boldsymbol{x}+\boldsymbol{\beta}; B) + \Theta(\boldsymbol{v}+\boldsymbol{\beta}; B)] + 2\langle \boldsymbol{v}, B^{-1}\boldsymbol{v}\rangle + 2\langle \boldsymbol{x}, B^{-1}\boldsymbol{x}\rangle \\
&= 2\max\left[\Theta(\boldsymbol{x}; B), \Theta\left(\boldsymbol{x}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) - \Theta(\boldsymbol{v}; B) - \Theta\left(\boldsymbol{v}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right)\right] + 2\langle \boldsymbol{x}, B^{-1}\boldsymbol{x}\rangle
\end{aligned}$$

where we use the assumption (9) and (13). On the other hand, the right-hand side of (2) is reduced to

$$\begin{aligned}
&\max\left[\rho^t_{n,m+1} + \rho^t_{n,m-1}, \rho^t_{n+1,m} + \rho^t_{n-1,m} - L\right] \\
&\quad = 2\max\Big[\Theta(\boldsymbol{x}; B), \Theta\left(\boldsymbol{x}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) - \Theta(\boldsymbol{w}; B) + \Theta\left(\boldsymbol{w}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right), \\
&\qquad \Theta(\boldsymbol{x}; B) + \Theta(\boldsymbol{u}; B) - \Theta\left(\boldsymbol{u}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) - \Theta(\boldsymbol{v}; B) + \Theta\left(\boldsymbol{v}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right), \\
&\qquad \Theta\left(\boldsymbol{x}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) - \Theta(\boldsymbol{v}; B) - \Theta\left(\boldsymbol{v}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right)\Big] + 2\langle \boldsymbol{x}, B^{-1}\boldsymbol{x}\rangle \\
&\quad = 2\max\left[\Theta(\boldsymbol{x}; B), \Theta\left(\boldsymbol{x}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right) - \Theta(\boldsymbol{v}; B) - \Theta\left(\boldsymbol{v}+\tfrac{\boldsymbol{1}}{\boldsymbol{2}}; B\right)\right] + 2\langle \boldsymbol{x}, B^{-1}\boldsymbol{x}\rangle
\end{aligned}$$

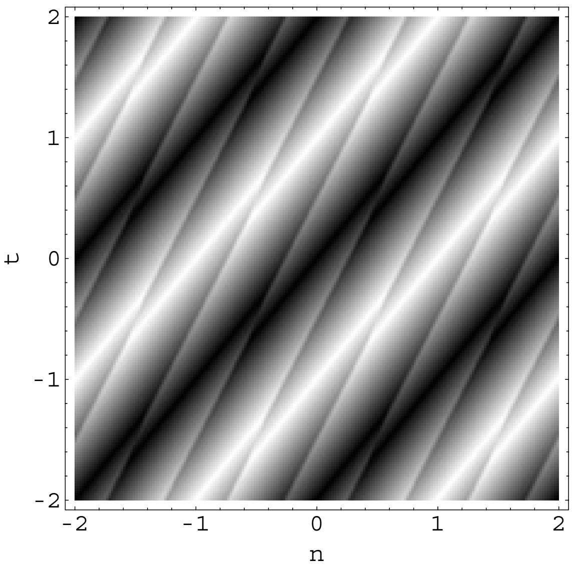

Figure 1. The periodic multiwave solution to the 1D TTCA with the choice of the parameter (19) on the nt-plane. A higher value is drawn in a brighter grey. There exist two periodic waves whose velocities are 1 and $\frac{1}{3}$, respectively. The phases of the periodic waves shift after the collision.

where we use the assumption (10)–(12) and (14). Thus, we conclude that if $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy the dispersion relations (9)–(14) then (8) solves (2). Similarly, if $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy the dispersion relations (15)–(17) then (8) solves (2). We obtain a periodic multiwave solution to the 2D TTCA from (8) through the relation (3).

Example 1. By setting $\boldsymbol{w} = 0$, we consider the 1D TTCA. Assume $g = 2$. Put

$$B = \begin{pmatrix} \frac{1}{5} & \frac{1}{11} \\ \frac{1}{11} & \frac{1}{13} \end{pmatrix} \qquad \boldsymbol{u} = \left(\tfrac{3}{2}, -\tfrac{1}{2}\right) \qquad \boldsymbol{v} = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{19}$$

Then B, $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy the dispersion relations (15)–(17) and we obtain a periodic multiwave solution U_n^t to the 1D TTCA through the relation (3) with eliminating m (see figure 1). The solution has periodic behaviour with respect to $n \to n + 2$ and $t \to t + 2$. We can see soliton-like behaviour of the periodic waves.

Now we show that we can obtain the solution (8) to the 2D TTCA form the one to the 2D d-Toda equation through the ultradiscretization. For any g, the procedure is essentially the same; substitute a solution expressed by $\vartheta(\boldsymbol{z}, \Omega)$ into the bilinear form of the 2D d-Toda equation and, by using an addition formula, obtain dispersion relations; then ultradiscretize the solution and the dispersion relations. Especially, for $g = 1$, the addition formula for $\vartheta(\boldsymbol{z}, \Omega)$ (resp. $\Theta(\boldsymbol{x}; B)$) is nothing but the bilinear form of the 2D d-Toda equation (resp. 2D TTCA). Therefore, we show the case $g = 1$.

Note an addition formula for elliptic theta function $\vartheta_{00}(z) := \vartheta_{00}(z, \tau)$

$$\vartheta_{00}(x+y)\vartheta_{00}(x-y)\vartheta_{00}(0)^2 - (\vartheta_{00}(x)^2\vartheta_{00}(y)^2 + \vartheta_{11}(x)^2\vartheta_{11}(y)^2) = 0. \tag{20}$$

Then an identity for $\vartheta_{00}(z)$ is reduced:

$$\sum_{i=0}^{2} D(y_{i+1}, y_{i+2})\vartheta_{00}(x+y_i)\vartheta_{00}(x-y_i) = 0 \tag{21}$$

where we put

$$D(a,b) := \vartheta_{00}(a)^2\vartheta_{11}(b)^2 - \vartheta_{11}(a)^2\vartheta_{00}(b)^2.$$

The subscripts of y are considered mod 3.

Let

$$f_{n,m}^t = \eta\, \mathrm{e}^{\frac{\pi \mathrm{i}}{\tau}x^2}\vartheta_{00}(x,\tau) \qquad \eta := \mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\tau^{\frac{1}{2}} \qquad x = \tilde{u}n + \tilde{v}t + \tilde{w}m. \tag{22}$$

By substituting (22) into (21), we see that if the following holds

$$\delta^2 = \exp\left(-\frac{2\pi\mathrm{i}}{\tau}(\tilde{u}^2 - \tilde{v}^2)\right)\frac{D(\tilde{w},\tilde{v})}{D(\tilde{w},\tilde{u})} \tag{23}$$

$$1 = \frac{\exp\left(\frac{2\pi\mathrm{i}}{\tau}\tilde{v}^2\right)}{D(\tilde{w},\tilde{u})}\left\{\exp\left(-\frac{2\pi\mathrm{i}}{\tau}\tilde{u}^2\right)D(\tilde{w},\tilde{v}) + \exp\left(-\frac{2\pi\mathrm{i}}{\tau}\tilde{w}^2\right)D(\tilde{v},\tilde{u})\right\} \tag{24}$$

then (22) solves the bilinear form of the 2D d-Toda equation (1).

Remark 3. Assume τ to be pure imaginary then there exist real solutions $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ to (24).

Remark 4. Assume $\tilde{w} = 0$ in (22)–(24) then a periodic single wave solution to the 1D d-Toda equation is reduced.

Now we ultradiscretize the identity (21). If we substitute $g = 1$ and $B = 1$ into $\Theta(\boldsymbol{x}; B)$, then we obtain the ultradiscrete elliptic theta function [8, 14]:

$$\Theta_0(x) := \Theta(x;1) = -x^2 + \max_{n\in\mathbb{Z}}[2nx - n^2].$$

Elliptic theta function $\vartheta_{11}(z)$ is ultradiscretized as follows [8]:

$$\lim_{\varepsilon\to+0}\varepsilon\log^{\dagger}\vartheta_{11}(z, \mathrm{i}\pi\varepsilon)|_{\mathrm{Im}\, z = \frac{\pi}{4}\varepsilon} = \Theta_0\left(\mathrm{Re}\, z + \tfrac{1}{2}\right) =: \Theta_{\frac{1}{2}}(\mathrm{Re}\, z).$$

By setting $g = 1$ in the addition formula (7), we obtain an addition formula for $\Theta_0(x)$:

$$\Theta_0(x+u) + \Theta_0(x-u) = 2\max\left[\Theta_0(x) + \Theta_0(u), \Theta_{\frac{1}{2}}(x) + \Theta_{\frac{1}{2}}(u)\right]. \tag{25}$$

Using the addition formula (25), we obtain an identity

$$\begin{aligned}\Theta_0(x+y_0) + \Theta_0(x-y_0) = \max\Big[&\Theta_0(x+y_1) + \Theta_0(x-y_1) + 2\Theta_{\frac{1}{2}}(y_0) - 2\Theta_{\frac{1}{2}}(y_1),\\ &\Theta_0(x+y_2) + \Theta_0(x-y_2) + 2\Theta_0(y_0) - 2\Theta_0(y_2)\Big]\end{aligned} \tag{26}$$

under an assumption

$$\Theta_0(y_1) - \Theta_{\frac{1}{2}}(y_1) < \Theta_0(y_0) - \Theta_{\frac{1}{2}}(y_0) < \Theta_0(y_2) - \Theta_{\frac{1}{2}}(y_2). \tag{27}$$

Actually, noting the inequality (27), the right-hand side of (26) is calculated as follows:

$$\begin{aligned}\text{rhs of (26)} &= \max\Big[2\Theta_0(x) + 2\Theta_0(y_1) + 2\Theta_{\frac{1}{2}}(y_0) - 2\Theta_{\frac{1}{2}}(y_1), 2\Theta_{\frac{1}{2}}(x) + 2\Theta_{\frac{1}{2}}(y_0),\\ &\qquad 2\Theta_0(x) + 2\Theta_0(y_0), 2\Theta_{\frac{1}{2}}(x) + 2\Theta_{\frac{1}{2}}(y_2) + 2\Theta_0(y_0) - 2\Theta_0(y_2)\Big]\\ &= \max\Big[2\Theta_{\frac{1}{2}}(x) + 2\Theta_{\frac{1}{2}}(y_0), 2\Theta_0(x) + 2\Theta_0(y_0)\Big].\end{aligned}$$

Therefore (25) leads (26).

Remark 5. The identity (26) is an ultradiscretization of the identity (21). Assume[1]

$$\left(\frac{\vartheta_{00}(y_1)}{\vartheta_{11}(y_1)}\right)^2 < \left(\frac{\vartheta_{00}(y_0)}{\vartheta_{11}(y_0)}\right)^2 < \left(\frac{\vartheta_{00}(y_2)}{\vartheta_{11}(y_2)}\right)^2. \tag{28}$$

Write (21) as follows:

$$\begin{aligned}\vartheta_{00}(x+y_0)\vartheta_{00}(x-y_0)\{\vartheta_{11}(y_1)^2\vartheta_{00}(y_2)^2 - \vartheta_{00}(y_1)^2\vartheta_{11}(y_2)^2\}\\ = \vartheta_{00}(x+y_1)\vartheta_{00}(x-y_1)\{\vartheta_{00}(y_2)^2\vartheta_{11}(y_0)^2 - \vartheta_{11}(y_2)^2\vartheta_{00}(y_0)^2\}\\ + \vartheta_{00}(x+y_2)\vartheta_{00}(x-y_2)\{\vartheta_{00}(y_0)^2\vartheta_{11}(y_1)^2 - \vartheta_{11}(y_0)^2\vartheta_{00}(y_1)^2\}.\end{aligned}$$

Then all the terms of the above equation are positive for $x, y_i \in \mathbb{R}$ $(i = 0, 1, 2)$. Therefore, we can ultradiscretize it and obtain (26).

Finally, we show that the solution (22) to the 2D d-Toda equation is ultradiscretized into the solution (8) to the 2D TTCA together with the dispersion relations. Let $\tau = \mathrm{i}\frac{\pi\varepsilon}{a}$ for $a, \varepsilon \in \mathbb{R}_{>0}$. Put

$$\rho^t_{n,m} = \lim_{\varepsilon\to+0} \varepsilon \log^\dagger f^t_{n,m}.$$

Then, we have

$$\rho^t_{n,m} = a\Theta_0(x) + ax^2 \qquad x = un + vt + wm.$$

This is the solution (8) with $g = 1$ and $B = \frac{1}{a}$. In this case, the dispersion relations (12), (13) and (14) vanish, (9) and (10) are reduced to the following:

$$\Theta_0(v) + v^2 = \Theta_0(w) + w^2 = 0 \tag{29}$$

and (11) is reduced to (27) with $\boldsymbol{u} = y_1$, $\boldsymbol{v} = y_0$ and $\boldsymbol{w} = y_2$. The relation (18) is reduced to

$$L = 2a\big(u^2 - v^2 + \Theta_{\frac{1}{2}}(u) - \Theta_{\frac{1}{2}}(v)\big). \tag{30}$$

Put $L = -\lim_{\varepsilon\to+0} \varepsilon \log^\dagger \delta^2$. Then the conditions (23) and (24) are ultradiscretized into (30) and (29) respectively.

4. Conclusion

We propose an ultradiscretization of the Riemann theta function and derive an addition formula. The addition formula is reduced from an ultradiscrete analogue of the generalized Riemann theta formula. Using the addition formula, periodic multiwave solutions to the 2D TTCA are obtained. The solutions are directly connected to the one to the 2D d-Toda equation through the ultradiscretization. Since the 2D d-Toda equation is equivalent to the discrete KP equation, the technique to obtain periodic multiwave solutions can be applied to several ultradiscrete integrable systems such as the box-ball system.

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[1] An ultradiscretization of (28) is (27).

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